Probability and Statistics for Engineers

Chapter 2: Probability

Lecturer

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Term 192

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Experiment and Outcomes

- By experiment we mean an act of conducting a controlled test or investigation.
- The experiment results in something.
- The possible results of an experiment may be one or more.

Definition (Deterministic experiment)

The experiment which has only one possible result or outcome (whose result is certain or unique) is called **deterministic or predictable experiments**.

Definition (Random experiment)

A **random experiment** is an experiment that can result in different outcomes, even though it is repeated in the same manner every time.

Definition (Outcome)

An **outcome** is the result of a single trial of a random experiment.

Random Experiment

- The goal is to understand, quantify and model the variation affecting a physical system's behavior.
- The model is used to analyze and predict the physical system's behavior as system inputs affect system outputs.
- The predictions are verified through experimentation with the physical system.

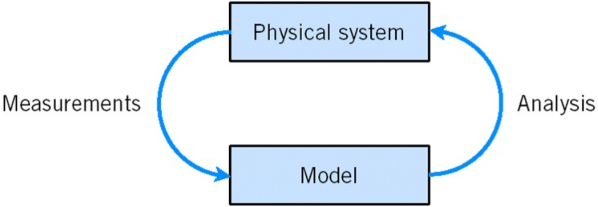


Figure: Continuous iteration between model and physical system

Random Experiment

- Random values of the noise variables cannot be controlled and cause the random variation in the output variables.
- Holding the controlled inputs constant does not keep the output values constant.

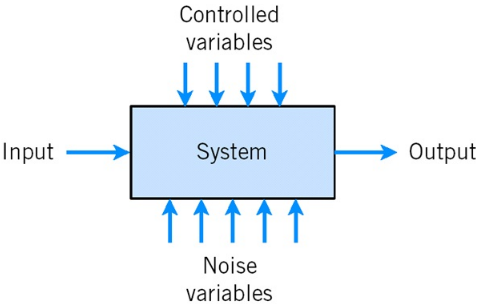


Figure: Noise variables affect the transformation of inputs to outputs

Random Experiment: Randomness Affects Natural Law

- Measuring the current in a thin copper wire, our model for the system might simply be Ohm's law.
- Because of uncontrollable inputs, variations in measurements of current are expected.
- Ohm's law might be a suitable approximation.
- However, if the variations are large relative to the intended use of the device under study, we might need to extend our model to include the variation.

Random Experiment: Randomness Can Disrupt a System

- Telephone systems must have sufficient capacity (lines) to handle a random number of callers at a random point in time whose calls are of a random duration.
- If calls arrive exactly every 5 minutes and last for exactly 5 minutes, only 1 line is needed - a deterministic system.
- Practically, times between calls are random and the call durations are random. Calls can come into conflict as shown in following slide.
- Conclusion: Telephone system design must include provision for input variation.

Random Experiment: Deterministic & Random Call Behavior

- Calls arrive every 5 minutes.
- In top system, call durations are all of 5 minutes exactly.
- In bottom system, calls are of random duration, averaging 5 minutes, which can cause blocked calls, a "busy" signal.

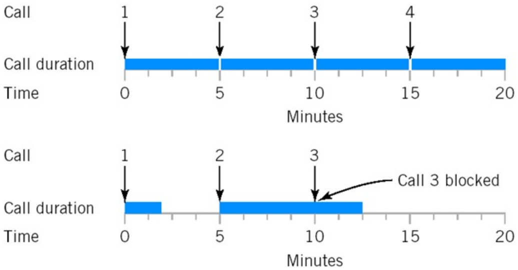


Figure: Variation causes disruptions in the system

Sample Space

Definition (Sample space)

The set of all possible outcomes of a random experiment is called the **sample space** and is denoted by S.

Definition (Discrete sample space)

A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes.

Definition (Continuous sample space)

A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.

Sample Space

Example

1. If the experiment consists of flipping a coin once, then the sample space consists of the following points:

2. If the experiment consists of flipping two **different** coins (or a coin twice), then the sample space consists of the following points:

3. If the experiment consists of rolling a die, then the sample space consists of the following points:

4. If the experiment consists of rolling two **different** dice, then the sample space consists of the following points:

Sample Space

Example

1. If the experiment consists of flipping a coin continuously until obtaining the first head, then the sample space is:

2. If the experiment consists of measuring the delay of a flight, then the sample space is:

3. If the experiment consists of choosing a point from the interval $(0, 1)$, then the sample space is:

4. If the experiment consists of selecting a random sample of size 3 from a class of 23 students, then the sample space is:

Events

Definition (Event)

An **event** is one or more outcomes of an experiment.

- Events will be denoted by Latin letters $A, B, C, \ldots$.
- The empty set, denoted by ϕ, is the subset of S containing no point. It is called an **impossible event** ("nothing occurs").
- The whole space S will be called a **sure or certain event** ("something occurs").
- If the outcome of a random experiment is member of an event, we say the event has occurred.

Events

Example

1. Consider the experiment of flipping two different coins. Let A be the event that a head appears on the first coin, then:

2. Consider the experiment of flipping two different coins. Let B be the event that a head appears once, then:

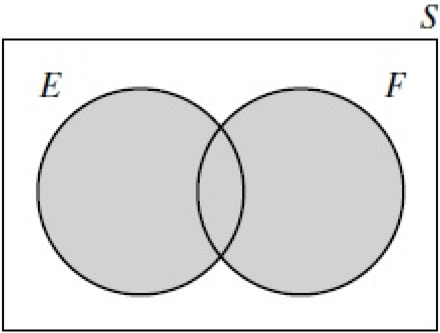
3. Consider the experiment of rolling two different dice. Let C be the event that the sum of the dice equals 7, then:

4. Consider the experiment of measuring the delay of a flight. Let D be the event that the flight departs within 1 hour of its scheduled departure, then:

Events: Venn Diagrams

For any two events E and F of a sample space S, one may define the following new events:

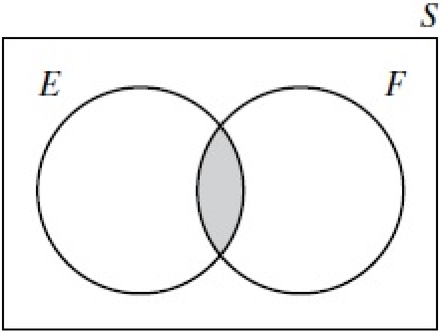
- **Union:** $E \cup F = \{$outcomes that are either in E or in F or in both E and $F\}$; that is, the event $E \cup F$ will occur if either E or F occurs.



Shaded region: $E \cup F$.

Events: Venn Diagrams

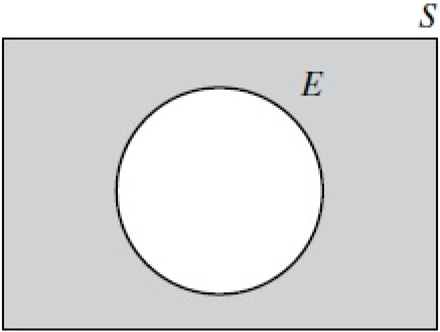
- **Intersection:** $E \cap F = \{$outcomes that are in both E and $F\}$; that is, the event $EF = E \cap F$ will occur only if both E and F occur.



Shaded region: EF.

Events: Venn Diagrams

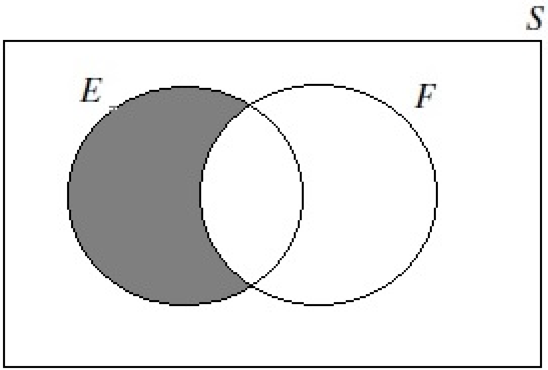
- **Complement:** $E^c = \{$outcomes in the sample space S that are not in $E\}$; that is, the event E^c will occur if and only if E does not occur.



Shaded region: E^c.

Events: Venn Diagrams

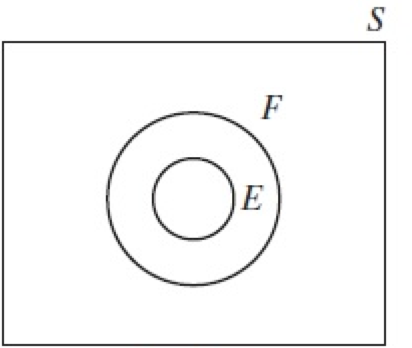
- **Difference:** $E - F = E \setminus F = E \cap F^c = \{$outcomes that are in E but not in $F\}$; that is, the event $E \setminus F$ occurs if E occurs but not F.



$E\text{-}F = E \cap F^c$

Events: Venn Diagrams

- **Subset:** An event E is a subset of F if whenever E occurs, F occurs; $E \subset F$.

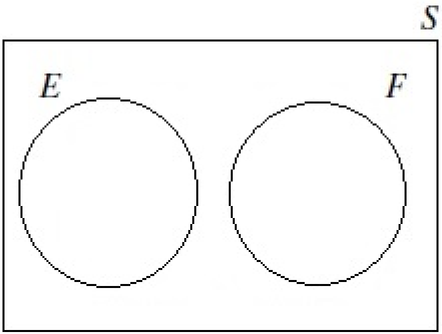


$E \subset F$

- **Equality:** $E = F$ if and only if $E \subset F$ and $E \supset F$.

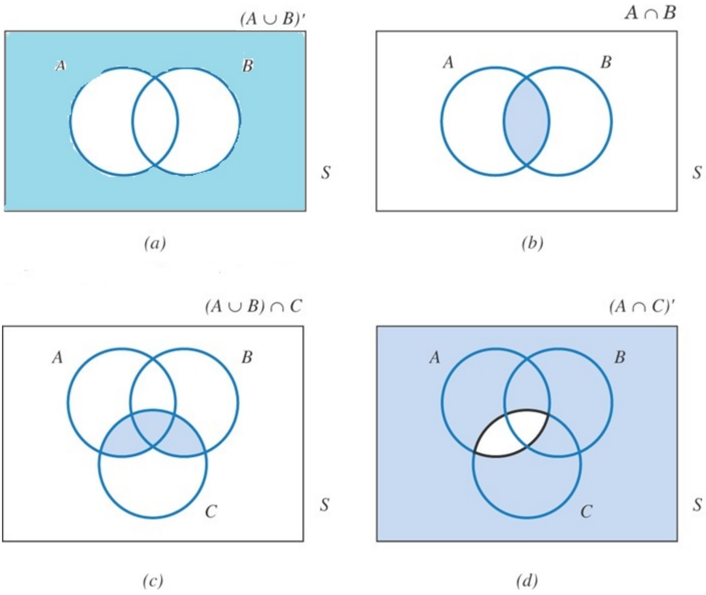
Events: Venn Diagrams

- **Mutually Exclusive Events:** Two events, E and F, are said to be mutually exclusive if $E \cap F = \phi$.



$$E \cap F = \phi$$

Events: Venn Diagrams



Events: Union and intersection

Unions and intersections of more than two events:

- **Union:** $E_1 \cup E_2 \cup \cdots = \bigcup_{i=1}^{\infty} E_i$: at least one of E_i occur.
- **Intersection:** $E_1 \cap E_2 \cap \cdots = \bigcap_{i=1}^{\infty} E_i$: all E_i occur.

Let A, B, and C be events in the same sample space, then

- **Commutative laws:**

$$A \cup B = B \cup A, \quad A \cap B = B \cap A.$$

- **Associative laws:**

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C).$$

- **Distributive laws:**

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \quad (A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Events: DeMorgan's law

DeMorgan's laws

1. $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$.

2. $\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$.

DeMorgan's laws: Two Events

1. $(E_1 \cup E_2)^c = E_1^c \cap E_2^c$.

2. $(E_1 \cap E_2)^c = E_1^c \cup E_2^c$.

Sample Space Is Defined By A Tree Diagram

Example

Messages are classified as on-time or late. 3 messages are classified. There are $2^3 = 8$ outcomes in the sample space.

$$S = \{ooo, ool, olo, oll, loo, lol, llo, lll\}.$$

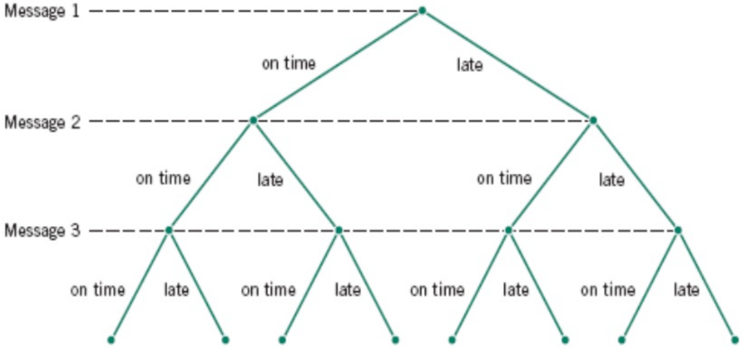


Figure: Tree diagram for three messages

Tree Diagrams Can Fit The Situation

Example

New cars can be equipped with selected options as follows:

1. Manual or automatic transmission
2. With or without air conditioning
3. Three choices of stereo sound systems
4. Four exterior color choices

Tree Diagrams Can Fit The Situation

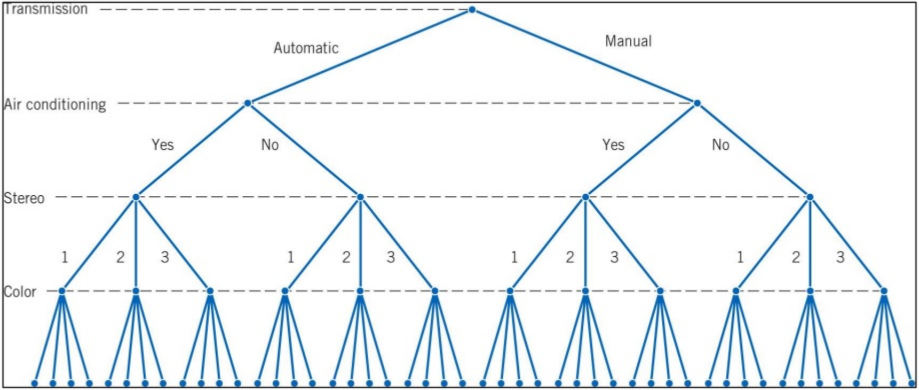


Figure: Tree diagram for different configurations of vehicles.

How many outcomes do we have in this diagram ?

Counting Techniques: The basic principle of counting (Multiplication Rule)

Assume an operation can be described as a sequence of k steps, and

- the number of ways of completing step 1 is n_1, and
- for each of these n_1, the number of ways of completing step 2 is n_2, and
- for each way of completing the first two steps, the number of ways of completing step 3 is n_3,
- and so forth.

The total number of ways of completing the operation is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Counting Techniques: The basic principle of counting

Example

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Example

Suppose we have 4 shirts of 4 different colors and 3 pants of different colors. How many possibilities are there?

Counting Techniques: The basic principle of counting

Example

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Example

Suppose a computer username consists of 3 **letters** followed by 5 **numbers**. How many usernames are possible?

What if no repetitions are allowed?

Permutations

Question
How many different ordered arrangements of the letters $a, b, \& c$ are possible?

Answer:
By direct enumeration we see that there are 6, namely,

$$abc, acb, bac, bca, cab, cba.$$

- Each arrangement is known as a permutation. Thus, there are 6 possible permutations of a set of 3 objects.
- This result could also have been obtained from the basic principle, since the **first object** in the permutation can be any of the 3, the **second object** in the permutation can then be chosen from any of the remaining 2, and the **third object** in the permutation is then the remaining 1. Thus, there are

$$3 \times 2 \times 1 = 6 \text{ possible permutations.}$$

Permutations

Definition (Permutations)

A permutation is an arrangement of objects in a certain order.

Fact

There are $n!$ (read as "n factorial") permutations of n distinct objects.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$$

- $3! = 3 \times 2 \times 1 = 6.$
- $0! = 1.$

Permutations

Example

How many different ways are there of shuffling a deck of cards?

Example

How many different ways can 6 different books be arranged on a shelf?

Example

How many ways to assign 5 hats to 5 people?

Permutations

Example

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

1. How many different rankings are possible?

2. If the men are ranked just among themselves and the women just among themselves,how many different rankings are possible?

Permutations

Example

How many ways can one arrange 4 math books, 3 chemistry books, 2 physics books, and 1 biology book on a bookshelf so that all the books dealing with the same subject are together on the shelf?

Solution

- *The number of ways of arranging the math books is: ... .*
- *The number of ways of arranging the chemistry books is: ... .*
- *The number of ways of arranging the physics books is: ... .*
- *The number of ways of arranging the biology book is: ... .*

The number of ways of arranging the subjects is: *... .*

Now, using the basic principle of counting, the number of arrangements is: ... .

Permutations

Example

A father, mother, 2 boys, and 3 girls are asked to line up for a photograph. Determine the number of ways they can line up if

1. there are no restrictions.
2. the parents stand together.
3. the parents do not stand together.
4. all the females stand together.

Permutations of Subsets

Definition (Permutations of Subsets)

The number of permutations of subsets of r elements selected from a set of n different elements is:

$$P_r^n = {}_nP_r = \frac{n!}{(n-r)!}.$$

Example

A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many different designs are possible?

Permutations

Example

Suppose you're given a list of 100 desserts and asked to rank your top 3. How many possible "top 3" lists are there?

Example

A student activity club at the college has 23 members. In how many different ways can the club select a president, a vice president, a treasurer, and a secretary?

Indistinguishable Permutations (Permutations of Similar Objects)

Definition (Indistinguishable objects)

Indistinguishable objects are simply items (letters) that are repeated in the original set.

For example

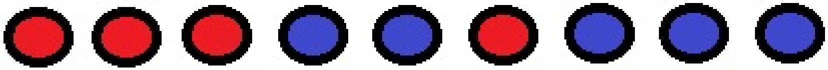


Figure: Indistinguishable Objects

- Assuming they are **distinguishable**, there are . . . possible arrangements.
- Some of these possible arrangements are repeated.
- How many times is this sequence repeated? Ans. $4! \times 5!$.
- Now, the number of **indistiguashable** arrangements are: $\frac{9!}{4! \times 5!}$

Indistinguishable Permutations (Permutations of Similar Objects)

Definition ((Permutations of Similar Objects))

The number of different permutations of n objects (not necessarily different), where there are:

- n_1 are of one type,
- n_2 are of a second type,
- ..., and
- n_r are of an rth type,

where $n_1 + n_2 + \ldots + n_r = n$, is:

$$\frac{n!}{n_1! \times n_2! \times \ldots \times n_r!}.$$

Indistinguishable Permutations (Permutations of Similar Objects)

Example

How many **different** letter arrangements can be formed from the letters PEPPER?

- $n = 6$ & $r = 3$.
- $n_1 = 3$.
- $n_2 = 2$.
- $n_3 = 1$.
- Hence, there are ... possible letter arrangements.

Example

Suppose there are 4 Czech tennis players, 4 U.S. players, and 3 Russian players, in how many ways could they be arranged?

Combinations

Definition (Combinations)

A **combination** is a selection of all or part of a set of objects, where the order of the selection is not important.

Principle

- The number of possible combinations of n taken r at a time (Sampling without replacement):

$$\binom{n}{r} = {}_nC_r = C_r^n = \frac{n!}{(n-r)!r!}, \quad r \leq n.$$

- ${}_nC_r$ is the number of different groups of size r that can be chosen from a set of size n objects (where the order of selection is not important).

- $\binom{n}{0} = \binom{n}{n} = 1.$

Combinations

Example

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Example

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

Combinations

Example

I am having a party, but I only have space for 5 of my 8 friends.

1. How many possible arrangements of guests friends do I have?

2. Suppose 2 are feuding and won't attend together. How many arrangements of guests are there now?

3. Suppose two of my friends will only come together. How many arrangements of guests are there now?

Combinations

Example

A bin of 50 manufactured parts contains three defective parts and 47 nondefective parts. A sample of six parts is selected from the 50 parts without replacement.

1. How many different samples are there of size 6 that can be chosen form the 50 parts?

2. How many different samples are there of size 6 that contain exactly two defective parts?

Combinations

Fact

$$\binom{n}{r} = \binom{n}{n-r}, \quad r \leq n.$$

Reason:
There are two ways of choosing a group of r objects out of n possible.

1. Choose r objects for the group: $\binom{n}{r}$.
2. Choose the $n - r$ objects left-out of the group: $\binom{n}{n-r}$.

Both methods give the same answer.

Definition (Probability)

Probability is used to **quantify** the likelihood, or chance, that an outcome of a random experiment will occur.

For example, "The chance of rain today is 30%", is a statement that **quantifies** our feeling about the possibility of rain.

- The likelihood of an outcome is quantified by assigning a number from the interval $[0, 1]$ (or a percentage from 0 to 100%) to the outcome.
- Higher numbers indicate that the outcome is more likely than lower numbers.
- A probability of 0 indicates an outcome will not occur.
- A probability of 1 indicates an outcome will occur with certainty.

Interpretations and Axioms of Probability

Interpretations of Probability

1. **Subjective approach.**
   Judgment is used as the basis for assigning probabilities (personal degree of belief). Ex's: Surgical operations, Horse race, ...

2. **Relative-frequency approach.**
   Consider long-run relative frequencies.

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}.$$

3. **Classical approach.**
   This method would assign a probability of $1/N$ to each outcome (equally likely outcomes).

$$P(E) = \frac{n(E)}{N}.$$

Interpretations and Axioms of Probability

A probability must satisfy the following three axioms (**Kolmogorov's Axioms**):

1. $P(S) = 1$.
2. $0 \leq P(E) \leq 1, \forall E \subset S$.
3. For two **mutually exclusive events** E_1 and E_2 ($E_1 \cap E_2 = \phi$), we have
$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

Definition (Mutually exclusive events)

Two events, E_1 and E_2 are said to be mutually exclusive if $E_1 \cap E_2 = \phi$.

Definition (Mutually exclusive events (More than two events))

$E_1, E_2, \ldots$ are said to be mutually exclusive if $E_i \cap E_j = \phi$, $\forall i \neq j$.

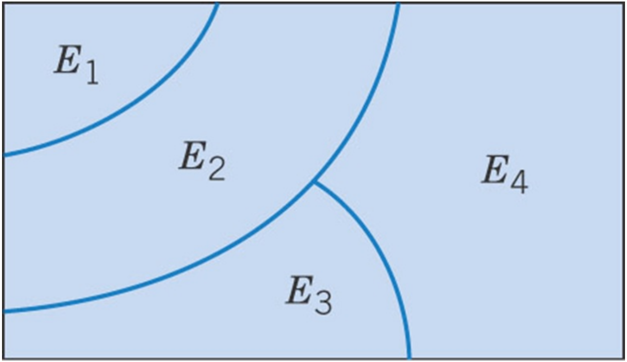


Figure: Venn diagram of four mutually exclusive events

Interpretations and Axioms of Probability

The probability axioms imply the following results:

- $P(\phi) = 0$.
- $P(E^c) = 1 - P(E)$.
- If $E \subset F$, then $P(E) \leq P(F)$.

Example

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$. Then

1. $P(A) =$
2. $P(B) =$
3. $P(C) =$
4. $P(A') = P(A^c) =$
5. $P(A \cap B) =$
6. $P(A \cup B) =$
7. $P(A \cap C) =$

Interpretations and Axioms of Probability

Example

Orders for a computer are summarized by the operational features that are requested as follows:

	proportion of orders
no optional features	0.2
one optional feature	0.6
more than one optional features	0.2

1. What is the probability that an order requests **at least one** optional feature?

2. What is the probability that an order does not request more than one optional feature?

Example

An injection-molded part is equally likely to be obtained from any one of the eight cavities on a mold.

1. What is the sample space?
2. What is the probability that a part is from cavity 1 or 2?
3. What is the probability that a part is from neither cavity 3 nor 4?

Interpretations and Axioms of Probability

Example

In an acid-base titration, a base or acid is gradually added to the other until they have completely neutralized each other. Because acids and bases are usually colorless (as are the water and salt produced in the neutralization reaction), pH is measured to monitor the reaction. Suppose that the equivalence point is reached after approximately 100 mL of a NaOH solution have been added (enough to react with all the acetic acid present) but that replicates are equally likely to indicate from 95 to 104 mL to the nearest mL. Assume that volumes are measured to the nearest mL.

1. What is the sample space?
2. What is the probability that equivalence is indicated at 100 mL?
3. What is the probability that equivalence is indicated at less than 100 mL?
4. What is the probability that equivalence is indicated between 98 and 102 mL (inclusive)?

Addition Rules: Probability of a Union

- For any two events E and F of the sample space S, we define

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

- For three events E_1, E_2, and E_3, we define

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) = & \; P(E_1) + P(E_2) + P(E_3) \\ & - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) \\ & + P(E_1 \cap E_2 \cap E_3). \end{aligned}$$

Addition Rules: Probability of a Union

Example

A single 6-sided die is rolled. What is the probability of rolling a number greater than 3 or an even number?

Example

In a math class of 30 students, 17 are boys and 13 are girls. On a unit test, 4 boys and 5 girls made an $A+$ grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an $A+$ student?

Addition Rules: Probability of a Union

Example

A single 6-sided die is rolled. What is the probability of rolling a 2 or a 5?

Example

A glass jar contains 1 red, 3 green, 2 blue, and 4 yellow marbles. If a single marble is chosen at random from the jar, what is the probability that it is yellow or green?

Conditional Probability

Suppose a fair die with six sides is rolled.

- The probability of rolling a four is ... .

- The probability of rolling an even number is ... .

The above probabilities are sometimes called **absolute or unconditional** probabilities.

The contrast is with **conditional** probabilities, which express the probability of one kind of event occurring **given that** another kind of event occurs.

- The conditional probability of rolling a four, given that an even number is rolled is ... .

Conditional Probability

Definition (Conditional Probability)

The conditional probability of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = \frac{A \cap B}{P(A)}, \quad P(A) > 0.$$

Example

Suppose a fair die with six sides is rolled. What is the probability of rolling a four, given that an even number is rolled?

Let $A = \{...\}$ and $B = \{...\}$, then $A \cap B = \{...\}$.

Therefore,

$$P(B|A) = \frac{...}{...}.$$

Conditional Probability

Example

At a middle school, 18% of all students play football and basketball and 32% of all students play football. What is the probability that a student plays basketball given that the student plays football?

Example

A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Conditional Probability

Example

At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?

Example

A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

Examples

Example

A lot of 100 semiconductor chips contains 25 that are defective. Two are selected randomly, without replacement, from the lot.

1. What is the probability that the first one selected is defective?
2. What is the probability that the second one selected is defective given that the first one was defective?
3. What is the probability that both are defective?

Examples

Example

Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		High	Low
scratch resistance	High	70	10
	Low	15	5

1. If a disk is selected at random, what is the probability that its scratch resistance is high (A) and its shock resistance is high (B)?
2. If a disk is selected at random, what is the probability that its scratch resistance is high (A) or its shock resistance is high (B)?
3. If a disk is selected at random and found that its shock resistance is low (E), what is the probability that its scratch resistance is high (A)?

Multiplication and Total Probability Rules

Definition (Multiplication Rule)

The probability that two events A and B both occur is given by the multiplication rule as:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example

A box contains 6 white balls and 4 red balls. We randomly (and without replacement) draw two balls from the box.

1. What is the probability that the first ball selected is white and the second ball is red?
2. What is the probability that the second ball selected is red?

Multiplication and Total Probability Rules

- A and A' are mutually exclusive events.
- $A \cap B$ and $A' \cap B$ are mutually exclusive.
- $B = (A \cap B) \cup (A' \cap B)$.

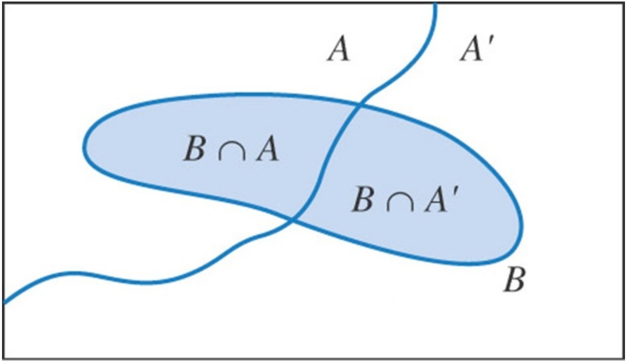


Figure: Partitioning an event into two mutually exclusive events

Multiplication and Total Probability Rules

Definition (Total Probability Rule)

For any events A and B,

$$\begin{aligned} P(B) &= P(A \cap B) + P(A' \cap B) \\ &= P(B|A)P(A) + P(B|A')P(A') \end{aligned}$$

Example

Suppose 1% of cotton fabric rolls and 2% of nylon fabric rolls contain flaws. Of the rolls used by a manufacturer, 70% are cotton and 30% are nylon. What is the probability that a randomly selected roll used by the manufacturer contains flaws?

1. Let B denote the event that a roll contains a flaw.
2. Let A denote the event that a roll is cotton.
3. Now, $P(B) = P(B|A)P(A) + P(B|A')P(A') = 0.013$.

Multiplication and Total Probability Rules

Example

The probability is 2% that an electrical connector that is kept dry fails during the warranty period of a portable computer. If the connector is ever wet, the probability of a failure during the warranty period is 6%. If 90% of the connectors are kept dry and 10% are wet, what **proportion (percentage)** of connectors fail during the warranty period?

1. Let B denote the event that a connector fails.
2. Let A denote the event that a connector is dry.
3. Now, $P(B) = P(B|A)P(A) + P(B|A')P(A') = 0.024$.

Multiplication and Total Probability Rules

Definition (Exhaustive Events)

A collection of sets $E_1, E_2, \dots, E_k$ are said to be **exhaustive** if

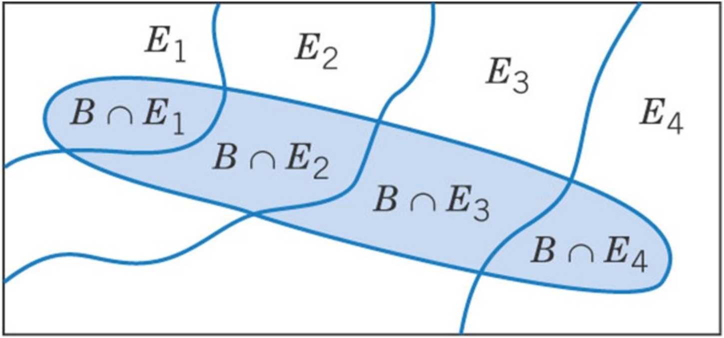
$$E_1 \cup E_2 \cup \dots \cup E_k = S$$

Definition (Total Probability Rule (Multiple Events))

Assume $E_1, E_2, \dots, E_k$ are k mutually **exclusive and exhaustive** events. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \end{aligned}$$

Multiplication and Total Probability Rules



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Figure: Partitioning an event into several mutually exclusive events

Example

The edge roughness of slit paper products increases as knife blades wear. Only 2% of products slit with new blades have rough edges, 3% of products slit with blades of average sharpness exhibit roughness, and 6% of products slit with worn blades exhibit roughness. If 25% of the blades in manufacturing are new, 60% are of average sharpness, and 15% are worn, what is the proportion of products that exhibit edge roughness?

1. Let B denote the event that a product exhibits edge roughness.
2. Let E_1, E_2, and E_3 denote the event that the blades are new, average, and worn, respectively.
3. Now, $P(B) = P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + P(B|E_3)P(E_3) = 0.032$.

Independence

Definition (Independence)

Two events, A and B, are said to be **independent** if any one of the following equivalent statements is true:

1. $P(A|B) = P(A)$.
2. $P(B|A) = P(B)$.
3. $P(A \cap B) = P(A)P(B)$.

Theorem

A and B are independent iff

1. *A' and B are independent.*
2. *A and B' are independent.*
3. *A' and B' are independent.*

Independence

Example

A batch of 600 containers for frozen orange juice contains 5 that are defective. Two are selected, at random, without replacement, from the batch. Let A and B denote the events that the first and second containers selected are defective, respectively.

1. Are A and B independent?

2. If the sampling were done with replacement, would A and B be independent?

Independence

Example

Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		High	Low
scratch resistance	High	70	10
	Low	16	4

Let A denote the event that a disk has high shock resistance, and let B denote the even that a disk has high scratch resistance.

1. Are events A and B independent?

2. Determine $P(B|A)$.

Bayes' Theorem

Example

Suppose urn B_1 has 2 red and 4 blue balls; urn B_2 has 1 red and 2 blue balls; and urn B_3 contains 5 red and 4 blue balls. Suppose the probabilities for selecting the urns are: $P(B_1) = \frac{1}{3}$, $P(B_2) = \frac{1}{6}$, and $P(B_3) = \frac{1}{2}$. A ball is drawn randomly,

1. Compute the probability of getting a red ball.

2. If the ball is red, what is the probability that it was from urn B_2?

Bayes' Theorem

Theorem (Bayes' Theorem)

*If $E_1, E_2, \ldots, E_k$ are k mutually **exclusive and exhaustive** events and B is any event, then*

$$
\begin{aligned}
P(E_i|B) &= \frac{P(B \cap E_i)}{P(B)} \\
&= \frac{P(B|E_i)P(E_i)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)}; \quad i = 1, 2, \ldots, k.
\end{aligned}
$$

Bayes' Theorem

Example

Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

1. What is the probability that a product attains a good review?

2. If a new design attains a good review, what is the probability that it will be a highly successful product?

Bayes' Theorem

Example

Suppose urn B_1 has 2 red and 4 blue balls; and urn B_2 has 3 red and 5 blue balls. A coin is tossed twice, if the outcome is $\{HH\}$ then a ball is drawn randomly from urn $B1$, otherwise the ball is drawn from urn B_2.

1. Compute the probability of getting a red ball.

2. If the ball is red, what is the probability that it was from urn B_2?

Random Variables

Definition (Random Variable)

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

$$X : S \to \mathbb{R}.$$

- For example, toss a coin three times and let X be the number of heads observed.

$$X = \{0, 1, 2, 3\}.$$

- A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70cm$.

Random Variables

Definition (Discrete Random Variable)

A **discrete random variable** is a random variable with a finite (or countably infinite) range.

Examples: number of scratches on a surface, number of transmitted bits received in error, number of students in each class of STAT319, etc.

Definition (Continuous Random Variable)

A **continuous random variable** is a random variable with an interval (either finite if infinite) or real numbers for its range.

Examples: electrical current, length, temperature, etc.