CALCULUS I

Chapter 2: Limits and Continuity

Lecturer

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Term 192

Table of Contents

Introduction

- **Calculus** is the study of functions and their properties.
- Three important topics of Calculus will be looked at this semester
  1. Limits of Functions;
  2. Continuity of Functions;
  3. Differentiation of Functions;
- Throughout the semester we will study each of these topics in some detail.
- There are many applications to each of these topics; these also will be surveyed.
- **Limit** is the most fundamental concept of topics (1)-(3) above. The concepts of continuity, differentiation, and integration all depend on the limit process. This will be our first topic of study.

Introduction

Motivating the concept of limit, we discuss three Fundamental Problem of Differential Calculus:

1. velocity;
2. slope of tangent lines;
3. rate of change of functions.

Rates of Change and Tangents to Curves

Objectives

1. To compute the average and instantaneous speed for objects.
2. To compute the average rate of change of a function (or the secant line).
3. To find the instantaneous rate of change of a function (or the slope of the tangent line to the curve of the function).

Rates of Change and Tangents to Curves

- In the late sixteenth century, **Galileo** discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling.

- This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling body.

- If y denotes the distance fallen after t seconds, then **Galileo's law** is

$$\begin{aligned} y &= 16t^2 \text{ feet} \\ &= 4.9t^2 \text{ meter,} \end{aligned}$$

  where 16 (or 4.9) is the (approximate) constant of proportionality.

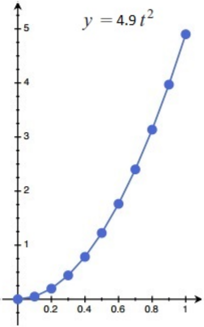
- A moving body's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed.

Rates of Change and Tangents to Curves

Drop a ball from the top of a building...

At time t, how far has the ball fallen?

time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900



Rates of Change and Tangents to Curves

- How fast is the ball falling at time t?

Definition (Average Speed)

The **average speed (velocity)** of the ball during a given time interval is the **change in distance**, Δy, divided by **the length of the time interval**, Δt.

- What is its average speed
  1. during the first 2 sec of fall? **Ans. 32 ft/sec.**
  2. during the 1-sec interval between second 1 and second 2? **Ans. 48 ft/sec.**

Rates of Change and Tangents to Curves

- If we need to find the **instantaneous velocity** at a given time, say t_0, we may take a small time interval $[t_0, t_0 + h]$ where h is "very small".

- The average velocity

$$\frac{\Delta y}{\Delta t} = \frac{f(t_0 + h) - f(t_0)}{h}, \quad y = f(t)$$

  will give us a good approximation of the velocity at t_0; the smaller we take h the better approximation we get.

- In fact, we define the **(instantaneous) velocity** at time t, of an object P, as

$$v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}.$$

Rates of Change and Tangents to Curves

- We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because **we cannot divide by zero**.

- But we can use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$.

- When we do so, we see a pattern.

Rates of Change and Tangents to Curves

TABLE 2.1 Average speeds over short time intervals

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16{t_0}^2}{h}$$

Length of time interval h	**Average speed over interval of length h starting at $t_0 = 1$**	**Average speed over interval of length h starting at $t_0 = 2$**
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

Rates of Change and Tangents to Curves

- The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases.

- This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec.

- Lets confirm this algebraically.
  - The average speeds over $[t_0, t_0 + h]$ is given by

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

  - If we set $t_0 = 1$ and then expand the numerator and simplify, we find that

$$\frac{\Delta y}{\Delta t} = 32 + 16h.$$

    It is clear that the average speed has the limiting value $32 + 16 \times 0 = 32$ ft/sec as h approaches 0.

  - Similarly, setting $t_0 = 2$ we get

$$\frac{\Delta y}{\Delta t} = 64 + 16h.$$

    As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec.

Rates of Change and Tangents to Curves

- The average speed of a falling object is just an example of a more general idea which we discuss now.
- Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$ by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs.

Definition (Average rate of change)

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$

Rates of Change and Tangents to Curves

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P\,(x_1, f(x_1))$ and $Q\,(x_2, f(x_2))$.

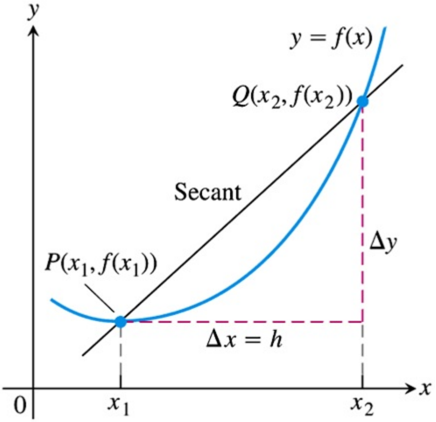
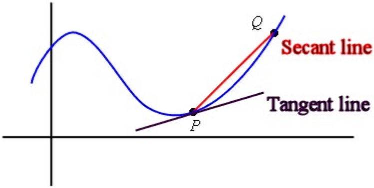


Figure: A secant to the graph $y = f(x)$. Its slope is $\frac{\Delta y}{\Delta x}$, the average rate of change of f over the interval $[x_1, x_2]$

Rates of Change and Tangents to Curves

- A secant line of a curve is a line that intersects two or more points on the curve.
- It can be used to approximate the tangent to a curve, at some point P.
- If the secant to a curve is defined by two points, P and Q, with P fixed and Q variable, as Q approaches P along the curve, the direction of the secant approaches that of the tangent at P.
- As a consequence, one could say that the limit of the secant's slope, or direction, is that of the tangent line at the point P.



Rates of Change and Tangents to Curves

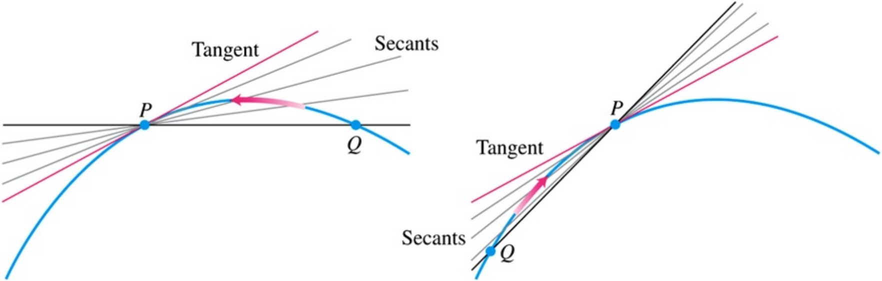


Figure: The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

Example

Find the slope of the curve $y = f(x) = 2x^2 + 1$ at the point $P(2, 9)$.

The slope of the secant line through the point $P(2, 9)$ and $Q\,(2 + h, f(2 + h))$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(2 + h) - f(2)}{(2 + h) - 2} = 8 + 2h.$$

As h gets closer and closer to 0 we see that the slope is 8.

Definition

The equation

$$y = y_1 + m(x - x_1)$$

is the point-slope equation of the line that passes through the point (x_1, y_1) and has slope m.

Rates of Change and Tangents to Curves

Example

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(2, 4)$.

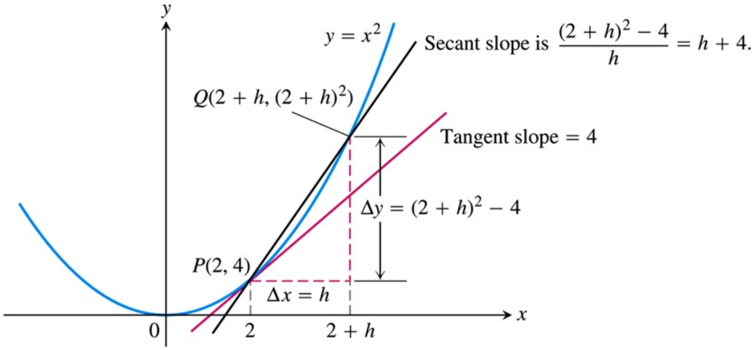


Figure: Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes

The tangent to the parabola at $P(2, 4)$ is the line through P with slope 4: $y = 4x - 4$.

Rates of Change and Tangents to Curves

Example

Find the average rate of change of the function $f(x) = 6x^3 + 6$ over the interval $[-6, 6]$.
Ans. 216

Example

Find an equation for the line tangent to $y = -5 - 7x^2$ at $(-3, -68)$. **Ans.** $y = 42x + 58$

Example

Find the slope of the curve $y = x^2 - 3x - 2$ at the point $P(2, -4)$ by finding the limit of the secant slopes through point P. Then find an equation of the tangent line to the curve at $P(2, -4)$. **Ans. 1,** $y = x - 6$

Limit of a Function and Limit Laws

Objectives

1. To explain the meaning of a limit.
2. To use properties of limits to calculate limits.
3. To apply limit laws to find one-sided limits.
4. To use the squeeze (sandwich or pinching) theorem to evaluate limits.

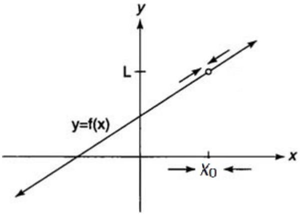
Limit of a Function and Limit Laws

- The limit of the function $f(x)$ as x approaches to x_0 is

$$\lim_{x \to x_0} f(x) = L, \quad L \in \mathbb{R}.$$

  Or,

$$f(x) \to L \text{ as } x \to x_0.$$



- $\lim_{x \to 3} f(x) = 6$ means as x approaches to 3, $f(x)$ approaches to 6.

- **One sided Limit:**

  1. Left-hand limit:
  $$\lim_{x \to x_0^-} f(x) = L,$$
  as x approaches x_0 from the left, $f(x)$ approaches L.

  2. Right-hand limit:
  $$\lim_{x \to x_0^+} f(x) = L.$$
  as x approaches x_0 from the right, $f(x)$ approaches L.

- In general, $\lim_{x \to x_0} f(x) = L$ iff $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$.

- If $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$, then $\lim_{x \to x_0} f(x)$ does not exist.

Limit of a Function and Limit Laws

- If f is the **identity function** $f(x) = x$, then for any value of x_0:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

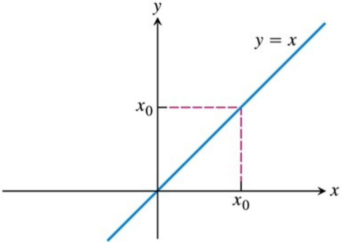


Figure: Identity function

- For example,

$$\lim_{x \to 3} x = 3.$$

Limit of a Function and Limit Laws

- If f is the **constant function** $f(x) = k$, then for any value of x_0:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

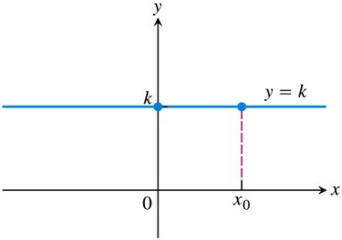


Figure: Constant function

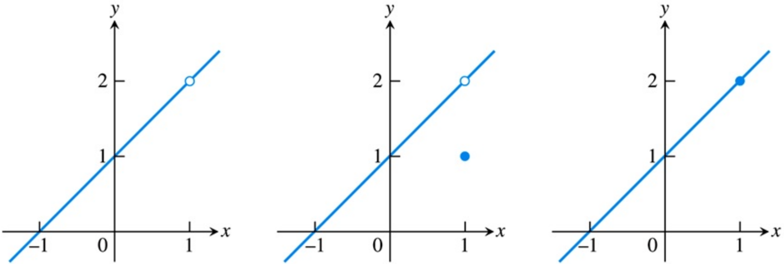
- For example,

$$\lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

Limit of a Function and Limit Laws

- The value of the function at the point x_0 is not important when we want to find the limit of the function at x_0.

- For example, consider the following functions:



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Figure: The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches $x_0 = 1$. However, only $h(x)$ has the same function value as its limit at $x = 1$

Limit of a Function and Limit Laws

- **Limits that do not exist**: limit may fail to exist at some points.
- For example, consider the following functions:

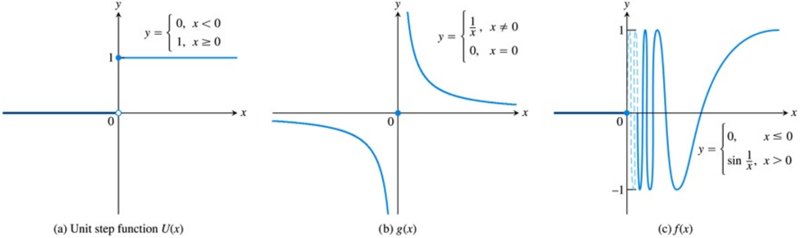


Figure: None of these functions has a limit as x approaches 0

- The function $f(x) = \sqrt{x-1}$ has no limit at 1 because it is not defined from the left of 1.

Limit of a Function and Limit Laws

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Limit of a Function and Limit Laws

Example

Evaluate the following limits:

1. $\lim_{x \to 2} \left(x^3 + 4x^2 - 3\right)$. **Ans.** 21

2. $\lim_{x \to 1} \left(\frac{x^4 + x^2 - 1}{x^2 + 5}\right)$. **Ans.** $\frac{1}{6}$

3. $\lim_{x \to -2} \sqrt{4x^2 - 3}$. **Ans.** $\sqrt{13}$

Limit of a Function and Limit Laws

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Limit of a Function and Limit Laws

Example

Evaluate the following limits:

1. $\lim_{x\to -1}\left(\frac{x^3+4x^2-3}{x^2+5}\right)$. **Ans.** $\frac{0}{6}=0$

2. $\lim_{x\to 0}\frac{1}{x^2}$. **Ans.** *DNE*

3. $\lim_{x\to 0}\frac{|x|}{x}$. **Ans.** *DNE*

Example

If $[\cdot]$ denotes the greatest integer function, let $f(x) = [x - 2]$ find,

1. Sketch the graph of $f(x)$.
2. $\lim_{x \to 2^+} f(x)$. **Ans.** 0
3. $\lim_{x \to 2^-} f(x)$. **Ans.** -1
4. $\lim_{x \to 2} f(x)$. **Ans.** *DNE*
5. $\lim_{x \to 3.1} f(x)$. **Ans.** 1

Indeterminate forms

- To evaluate limits of the form $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ we may substitute directly, and if the substitution leads to $\frac{0}{0}$, we say that the form is **indeterminate**.

- To handle these cases, we need further manipulations, such as **algebraic manipulation** or use other techniques.

- Some examples of these algebraic manipulations are the following.

Limit of a Function and Limit Laws

Example

Evaluate the following limits:

1. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$. **Ans.** 3

2. $\lim_{x\to 1} \frac{x-1}{x^3-1}$. **Ans.** $\frac{1}{3}$

   Recall: $(u^3 - v^3) = (u - v)(u^2 + uv + v^2)$.

3. $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$. **Ans.** 4

4. $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$. **Ans.** $\frac{1}{20}$

Limit of a Function and Limit Laws

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

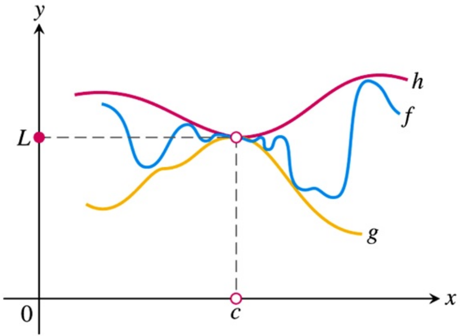


Figure: The graph of f is sandwiched between the graphs of g and h

Limit of a Function and Limit Laws

Example

Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$, find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

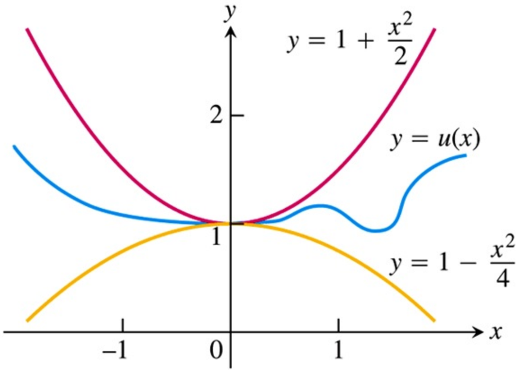


Figure: Any function $u(x)$ whose graph lies in the region between $y = 1 + \frac{x^2}{2}$ and $1 - \frac{x^2}{4}$ has limit 1 as $x \to 0$

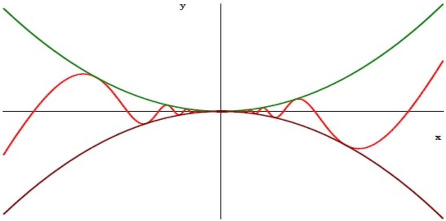
Limit of a Function and Limit Laws

Example

Evaluate the limit if exists:

1. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

   Recall: the sine function is always bounded in absolute value by 1, so $-1 \le \sin\left(\frac{1}{x}\right) \le 1$.



2. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$.

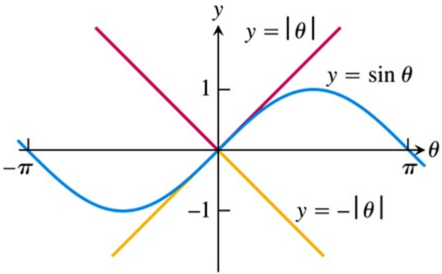
Example

The Sandwich Theorem (also called the Squeeze Theorem or the Pinching Theorem) helps us to establish several important limit rules:

- $\lim_{\theta \to 0} \sin \theta = 0$.

  Recall:

$$-|\theta| \leq \sin \theta \leq |\theta| \text{ for all } \theta.$$



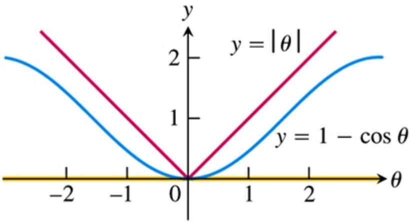
Limit of a Function and Limit Laws

Example

- $\lim_{\theta \to 0} \cos \theta = 1.$

  Recall:

$$0 \leq 1 - \cos \theta \leq |\theta| \text{ for all } \theta.$$



Limit of a Function and Limit Laws

Example

- If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

  Recall: for any function $f(x)$ we have

$$-|f(x)| \le f(x) \le |f(x)|.$$

THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

The Precise Definition of a Limit

Objectives

1. To define the limit using $\epsilon - \delta$ definition.
2. To find δ algebraically for given ϵ.

The Precise Definition of a Limit

Example

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x-1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

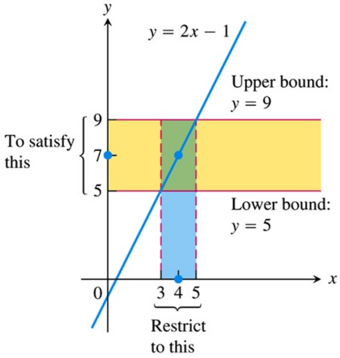


Figure: Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$

The Precise Definition of a Limit

Informal Definition: The limit of a function $f(x)$ is L as x approaches x_0, written $\lim_{x\to x_0} f(x) = L$, means that as x gets closer and closer (but not equal) to x_0 , the values of $f(x)$ get closer and closer to L. That is, we can make $f(x)$ arbitrarily close to L by choosing x very close to x_0. If no such $L \in \mathbb{R}$ exist, we say that the limit does not exist (DNE).

Definition (The Precise Definition of a Limit)

Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L, and write

$$\lim_{x\to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

The Precise Definition of a Limit

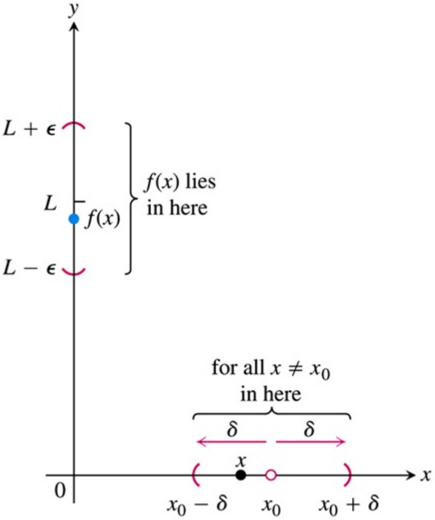


Figure: The relation of δ and ϵ in the definition of limit

The Precise Definition of a Limit

$\lim_{x \to x_0} f(x) = L$ means:

- You can get $f(x)$ as **close as you want** (ϵ) to L by making x sufficiently close to x_0.

- You tell me how close you want $f(x)$ to be to L (give positive number ϵ), I will find another number δ where if x is within δ of x_0, then $f(x)$ will be within ϵ of L.

- Given $\epsilon > 0$, we can find $\delta > 0$ such that if

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The Precise Definition of a Limit

Example

Use $\epsilon - \delta$ definition to prove that $\lim_{x\to 1}(5x - 3) = 2$.

1. For a any given $\epsilon > 0$, we need to find a $\delta > 0$ so that the following will be true:

$$|(5x - 3) - 2| < \epsilon \quad \text{whenever} \quad |x - 1| < \delta.$$

2. The first inequality can be simplified a little.

$$5|x - 1| < \epsilon.$$

3. So, the definition of the limit requires that,

$$|x - 1| < \frac{\epsilon}{5} \quad \text{whenever} \quad |x - 1| < \delta.$$

4. Therefore,

$$|(5x - 3) - 2| < \epsilon \quad \text{iff} \quad |x - 1| < \frac{\epsilon}{5}.$$

5. Taking $\delta = \frac{\epsilon}{5}$ (or any other smaller number) proves that $\lim_{x\to 1}(5x - 3) = 2$.

The Precise Definition of a Limit

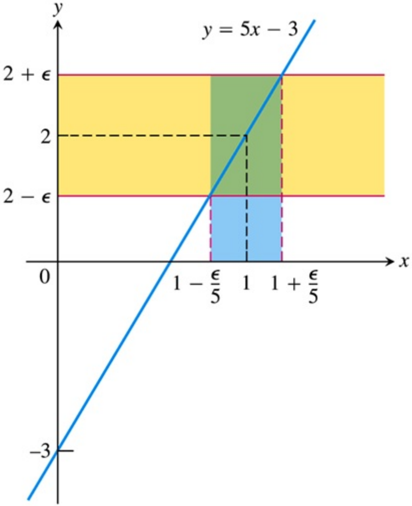


Figure: If $f(x) = 5x - 3$, then $0 < |x - 1| < \frac{\epsilon}{5}$ guarantees that $|(5x - 3) - 2| < \epsilon$

The Precise Definition of a Limit

Example

Use $\epsilon - \delta$ definition to prove that $\lim_{x \to 0} x^2 = 0$.

1. For a any given $\epsilon > 0$, we need to find a $\delta > 0$ so that the following will be true:

$$|x^2 - 0| < \epsilon \quad \text{whenever} \quad |x - 0| < \delta.$$

2. The first inequality can be simplified a little.

$$|x^2| = |x|^2 < \epsilon.$$

3. So, the definition of the limit requires that,

$$|x| < \sqrt{\epsilon} \quad \text{whenever} \quad |x| < \delta.$$

4. Therefore,

$$|x^2 - 0| < \epsilon \quad \text{iff} \quad |x - 0| < \sqrt{\epsilon}.$$

5. Taking $\delta = \sqrt{\epsilon}$ (or any other smaller number) proves that $\lim_{x \to 0} x^2 = 0$.

The Precise Definition of a Limit

Example

Use the graph of $f(x) = \frac{1}{x}$ to find the largest number of $\delta > 0$ such that if $|x - 1| < \delta$ then $|f(x) - 1| < 0.1$.

- $L + \epsilon = \frac{11}{10}$ and $L - \epsilon = \frac{9}{10}$.
- $f(x_1) = \frac{9}{10} \Longrightarrow x_1 = \frac{10}{9}$.
- $f(x_2) = \frac{11}{10} \Longrightarrow x_2 = \frac{10}{11}$.
- Then, $\delta_1 = \left|\frac{10}{9} - 1\right| = \frac{1}{9}$, and $\delta_2 = \left|1 - \frac{10}{11}\right| = \frac{1}{11}$.
- Therefore, $\delta = \min\{\delta_1, \delta_2\} = \frac{1}{11}$.

The Precise Definition of a Limit

Example

Use the graph of $f(x) = \sqrt{x}$ to find the largest number of δ such that $|\sqrt{x} - 2| < 0.4$ whenever $0 < |x - 4| < \delta$. **Ans.** $\delta = 1.44$

Example

For $\lim_{x \to 5} 2\sqrt{x+1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find $\delta > 0$ such that if $0 < |x - 5| < \delta$ then $|\sqrt{x-1} - 2| < 1$. **Ans.** $0 < \delta \leq 3$

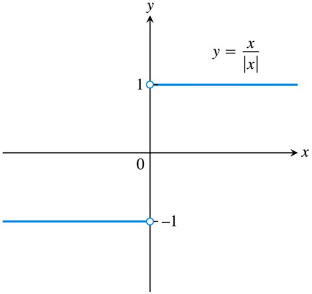
One-Sided Limits

Objective

1. To apply limit laws to find one-sided limits.

One-Sided Limits

- We know that some functions has no limits at a point.
- For example, the function $f(x) = \frac{x}{|x|}$ has no limit at 0 because it does not approach a real number at 0; i. e. $\lim_{x\to 0} f(x)$ does not exist. But it is clear that as x approaches 0 from the right the values of the function are approaching 1, while if x approaches 0 from the left the values of the function are approaching -1.



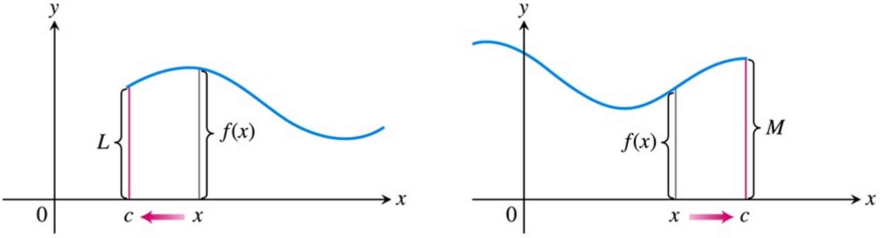
- Which means that the limit of the function may exist from one side of the point x_0.
- So, we may define the limit of the function from one side.

One-Sided Limits

Informal Definitions:

1. If $f(x)$ is defined on an interval (c, b) and approaches a number L as x approaches c from within that interval (from the right), then f has a right-hand limit L at c, and we write
$$\lim_{x \to c^+} f(x) = L.$$

2. Similarly, if $f(x)$ is defined on an interval (a, c) and approaches a number M as x approaches c from within that interval (from the left), then f has a left-hand limit M at c, and we write
$$\lim_{x \to c^-} f(x) = M.$$

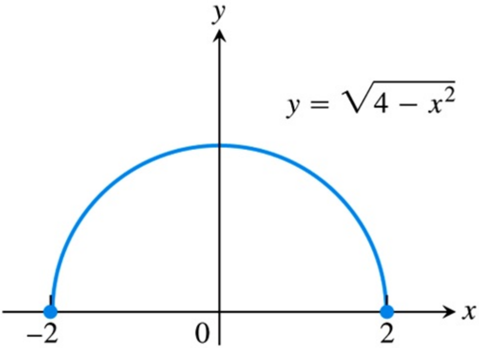


(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

One-Sided Limits

Example

Let $f(x) = \sqrt{4 - x^2}$. Find $\lim_{x \to -2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$.



$$\lim_{x \to 2^-} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to -2^+} \sqrt{4 - x^2} = 0$$

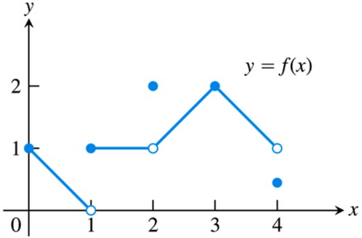
One-Sided Limits

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Example

Consider the function $f(x)$ whose graph is give in the figure.



Find: $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$.

One-Sided Limits

Definition

- We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x \to x_0^+} f(x) = L$$

  if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

  Or
$$0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

- We say that $f(x)$ has left-hand limit L at x_0, and write

$$\lim_{x \to x_0^-} f(x) = L$$

  if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

  Or
$$0 < x_0 - x < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

One-Sided Limits

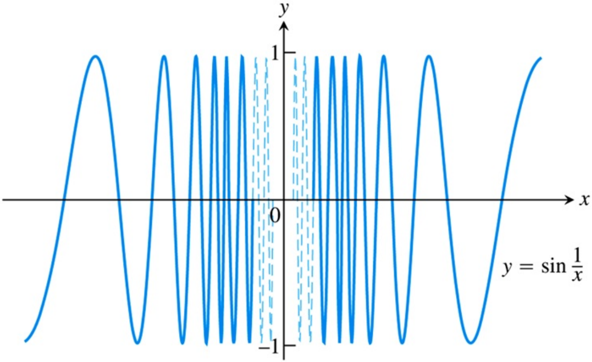
Example

Prove that $\lim_{x \to 1^+} \sqrt{x-1} = 0$.

- Given $\epsilon > 0$, we want to find $\delta > 0$ such that if $0 < x - 1 < \delta$ then $|\sqrt{x-1} - 0| < \epsilon$.
- $|\sqrt{x-1} - 0| < \epsilon$ iff $0 < \sqrt{x-1} - 0 < \epsilon$ iff $0 < \sqrt{x-1} < \epsilon$ iff $0 < x - 1 < \epsilon^2$.
- Thus, taking $\delta = \epsilon^2$ proves that $\lim_{x \to 1^+} \sqrt{x-1} = 0$.

One-Sided Limits

- We have seen before that the limit does not always exist.
- Similarly, the one sided limit may not exist also.
- For example, $\lim_{x\to 0^+} \sin\frac{1}{x}$ does not exist and also $\lim_{x\to 0^-} \sin\frac{1}{x}$, since the graph oscillates between -1 and 1 as you can see in the figure.



One-Sided Limits

Limits Involving $\frac{\sin\theta}{\theta}$.

- If we look at the graph of $\frac{\sin\theta}{\theta}$, we see that the values of the function tend to 1 as x approaches 0.

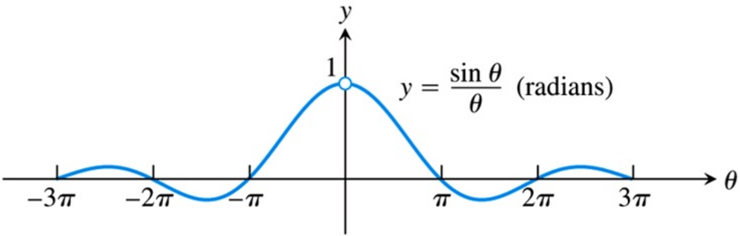


Figure: The graph of $f(\theta) = \frac{\sin\theta}{\theta}$ suggest that the right- and left-hand limits as θ approaches 0 are both 1

- In fact we have the following theorem.

One-Sided Limits

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

Example

Evaluate the following limits if exist.

1. $\lim_{x \to 0} \frac{1 - \cos x}{x}$. **Ans.** 0
2. $\lim_{x \to 0} \frac{\sin 2x}{5x}$. **Ans.** $\frac{2}{5}$
3. $\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$. **Ans.** $\frac{1}{3}$

One-Sided Limits

Example

Evaluate the following limits if exist.

1. $\lim_{x\to 0} \frac{\sin 3x}{x}$. **Ans.** 3

2. $\lim_{x\to 0} \frac{2x - \sin x}{x}$. **Ans.** 1

3. $\lim_{x\to 0} x \cot x$. **Ans.** 1

4. $\lim_{t\to 0} \frac{\sin 3x}{\sin 2x}$. **Ans.** $\frac{3}{2}$

5. $\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$. **Ans.** 0

Continuity

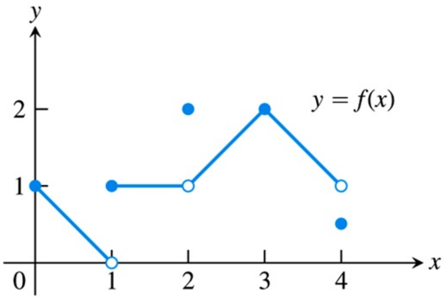
Objective

1. To define the continuous function at a number *a* and on an interval.
2. To examine the conditions of continuity from one side and from both sides.
3. To recognize the number of discontinuity and its type.
4. To apply some laws to show continuity.
5. To use the Intermediate Value Theorem (IVT) as a property of continuity.

Continuity

Example

Find the points at which the function f is continuous and the points at which f is not continuous.



The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1$, $x = 2$, and $x = 4$.

Continuity

Points at which f is continuous:

At $x = 0$,	$\lim_{x \to 0^+} f(x) = f(0)$.
At $x = 3$,	$\lim_{x \to 3} f(x) = f(3)$.
At $0 < c < 4$, $c \neq 1, 2$,	$\lim_{x \to c} f(x) = f(c)$.

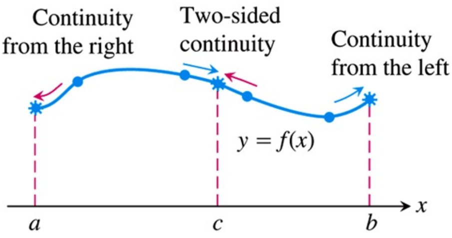
Points at which f is not continuous:

At $x = 1$,	$\lim_{x \to 1} f(x)$ does not exist.
At $x = 2$,	$\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.
At $x = 4$,	$\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.
At $c < 0$, $c > 4$,	these points are not in the domain of f.

Continuity

Definition

1. **Interior Point**: A function $y = f(x)$ is continuous at an interior point c of its domain if
$$\lim_{x \to c} f(x) = f(c).$$
2. **End Point**: A function $y = f(x)$ is continuous at a left endpoint a or is continuous at a right endpoint b of its domain if
$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \text{ respectively.}$$



- If f is not continuous at c , it is said to be **discontinuous** at c.

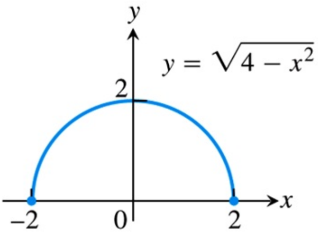
- Also we say that f is **left continuous** if it is continuous from the left, and it is **right continuous** if it is continuous from the right at c.

- Note that f is continuous at c if it is left and right continuous at c.

Continuity

Example

Consider the function $f(x) = \sqrt{4 - x^2}$.

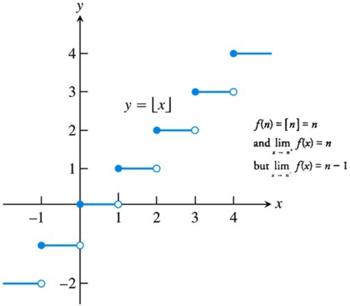


- It is continuous at every point of the interval $(-2, 2)$.
- It is left continuous at 2.
- It is right continuous at -2.
- Therefore, it is continuous at every point of its domain $[-2, 2]$.

Continuity

Example

Consider the function $f(x) = [x]$.



1. It is discontinuous at the integers ($\lim_{x\to n^+} f(x) \neq \lim_{x\to n^-} f(x)$).
2. It is right continuous at the integers ($\lim_{x\to n^+} f(x) = f(n) = n$).
3. It is not left continuous at the integers ($\lim_{x\to n^-} f(x) = n - 1 \neq f(n) = n$).

Continuity

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

If any of the three conditions is NOT satisfied at a number b, then b is called a number of discontinuity of the function f.

Continuity

Example

Find all values of A and B which will make f continuous everywhere.

$$f(x) = \begin{cases} x^2 - A & \text{if } x < 1 \\ A + Bx & \text{if } 1 \leq x \leq 2 \\ B - x^2 & \text{if } x > 2 \end{cases},$$

For f to be continuous everywhere we need it to be continuous at $x = 1$ and $x = 2$.

- At $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$ implies $2A + B = 1$.
- At $x = 2$: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ implies $A + B = -4$.
- $A = 5$ and $B = -9$.

Continuity

Types of Discontinuities

1. **Removable discontinuity**: if $\lim_{x\to a} f(x)$ exists at a but f is discontinuity at a. (may be not defined or defined outside the graph)

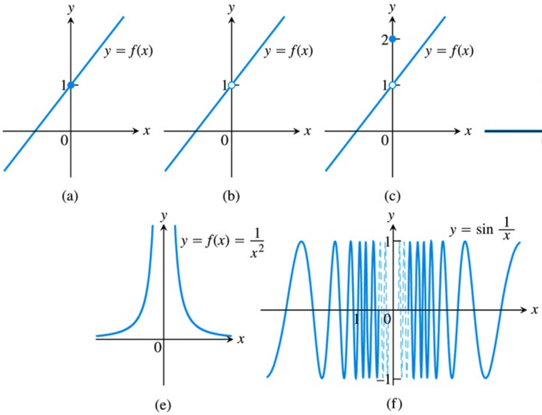
2. **Jump discontinuity**: if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist but are not equal.

3. **Infinite discontinuity**: if the left- and right-hand limits are infinite; they may be both positive, both negative, or one positive and one negative.

4. **Oscillating discontinuity**: if $\lim_{x\to a} f(x)$ DNE because f oscillates too much close to a.

Continuity

Example



The function in figure (a) is continuous, in (b) and (c) has removable discontinuity at 0, in (d) has jump discontinuity, in (e) has infinite discontinuity and in (f) has oscillating discontinuity.

Continuity

Example

Let $f(x) = \frac{\sqrt{2x+9}-\sqrt{x+9}}{2x}$. What kind of discontinuity does f have at $x = 0$? Justify.

Example

Let

$$f(x) = \begin{cases} x^2 - 1 & \text{if } -1 \leq x < 0 \\ 2x & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \\ -2x + 4 & \text{if } 1 < x < 2 \\ 0 & \text{if } 2 < x < 3 \end{cases},$$

Is f continuous at $x = 2$? If it is not continuous, what type of discontinuity does it have?

Continuity

Definition (Continuous Functions)

A function f is said to be continuous on an interval, if it is continuous at every point of the interval.

Example

1. Consider the function $f(x) = \frac{1}{x}$.
   - It is a continuous function because it is continuous at every point of its domain $(-\infty, 0) \bigcup (0, \infty)$.
   - The point $x = 0$ is a point of discontinuity of f.
   - It is discontinuous on any interval containing $x = 0$.
2. The identity function $f(x)$ and constant functions are continuous everywhere.

Continuity

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Continuity

The following functions are continuous on every number in their domain.

- Polynomials,
- Rational functions,
- Root functions,
- Trigonometric functions,
- Inverse trigonometric functions,
- Exponential functions,
- Logarithmic functions.

Continuity

Example

- The polynomial $f(x) = 3x^3 + x^2 - 1$ is continuous on $(-\infty, \infty)$.
- The rational function $\frac{P(x)}{Q(x)}$ is continuous at every number a where $Q(a) \neq 0$. For example, $f(x) = \frac{x^2+1}{x-1}$ is continuous everywhere except at $x = 1$.
- The function $f(x) = |x - 1|$ is continuous everywhere.
- The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.
- The functions $f(x) = \sin x$ and $g(x) = \cos x$ are continuous on $(-\infty, \infty)$.
- The function $f(x) = \tan x$ is continuous on all real numbers except when $x = n\pi + \frac{\pi}{2}$, where n is an integer.

Continuity

Example

If f and g are continuous at $x = 3$ with $f(3) = 5$, $\lim_{x\to 3}[2f(x) - g(x)] = 4$. Find $g(3)$.
Ans. $g(3) = 6$

Remark

The inverse $f^{-1}(x)$ of a continuous function is continuous on its domain. The reason is that the graph of the inverse function is obtained by reflecting the graph of $f(x)$ about the line $y = x$. Thus if the graph of $f(x)$ does not have a break, so does the graph of $f^{-1}(x)$.

Continuity

Example

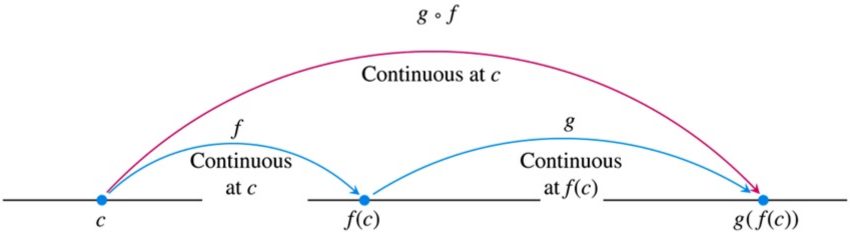
Where is $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

- $\ln x$ is continuous for all $x > 0$.
- $\tan^{-1} x$ is continuous for all $x \in \mathbb{R}$.
- $x^2 - 1$ is continuous for all $x \in \mathbb{R}$.
- $f(x)$ is not defined on $(-\infty, 0] \bigcup \{1\}$.

Therefore, $f(x)$ is continuous on $(0, 1) \bigcup (1, \infty)$.

Continuity

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



Continuity

Example

Show that the following functions are continuous on their domains.

1. $f(x) = \sqrt{x^2 - 2x - 5}$.

2. $f(x) = \left|\frac{x-2}{x^2-2}\right|$.

3. $f(x) = \left|\frac{x \sin x}{x^2+2}\right|$.

4. $f(x) = \ln(1 + \sin x)$.

Continuity

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

Example

Evaluate the following limits.

1. $\lim_{x\to\frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$. **Ans.** -1

2. $\lim_{x\to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right)$. **Ans.** $\frac{\pi}{6}$

3. $\lim_{x\to 0} \sqrt{x+1}e^{\tan x}$. **Ans.** 1

Continuity

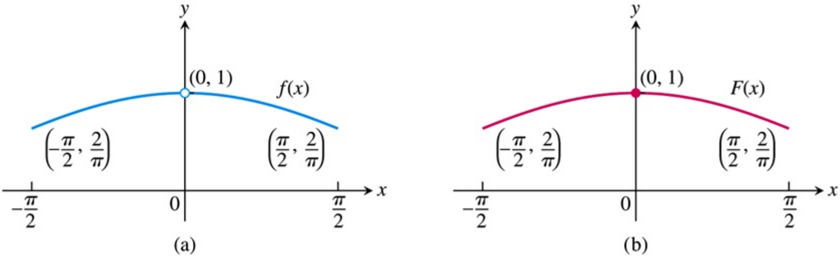
Continuous extension to a point:

- The function $f(x) = \frac{\sin x}{x}$ is continuous at every point except $x = 0$.
- It is like the function $f(x) = \frac{1}{x}$.
- But the function $\frac{\sin x}{x}$ has a finite limit as $x \to 0$.
- It is therefore possible to extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$.
- We define a new function

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

- Now, the function $F(x)$ is continuous at $x = 0$ because $\lim_{x\to 0} \frac{\sin x}{x} = F(0)$.

Continuity


y
(0, 1)
f(x)
$\left(-\frac{\pi}{2}, \frac{2}{\pi}\right)$
$\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$
x
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
(a)
y
(0, 1)
F(x)
$\left(-\frac{\pi}{2}, \frac{2}{\pi}\right)$
$\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$
x
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
(b)

Continuity

Definition

If $\lim_{x \to c} f(x) = L$ exists but $f(c)$ is not defined, then the function

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ \\ L & \text{if } x = c \end{cases},$$

is called the continuous extension of $f(x)$ to $x = c$.

Remark

We may extend rational function $f(x)$ to make it continuous at c by canceling common factors $(x - c)$ from the numerator and denominator.

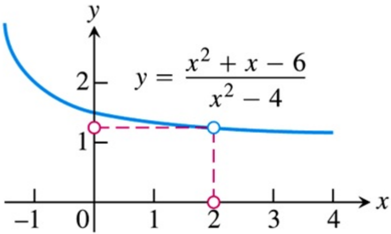
Continuity

Example

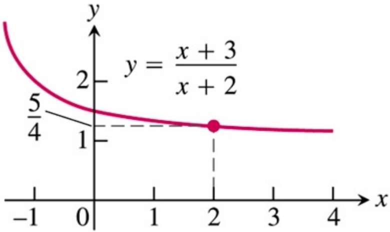
Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$ and find that extension.

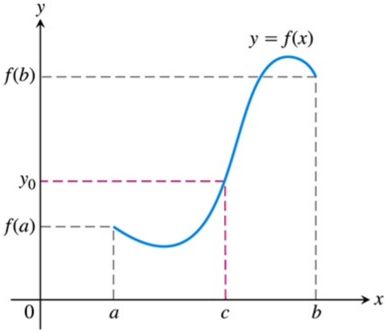


(a)



(b)

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Continuity

- A function is said to have the Intermediate Value Property if whenever it takes on two values, it also takes on all the values in between.

- The IVT says that the graph of a continuous function on a closed interval cannot have a break over that interval.

- Geometrically, the IVT says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

- If f is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of $f(x) = 0$ in the interval (a, b).

Continuity

Example

Consider the function $f(x) = x^2 - 1$ on the interval $[-1, 3]$.

- $f(-1) = 0$ and $f(3) = 8$.
- The number 2 is between 0 and 8.
- We need to find a number x between -1 and 3 such that $f(x) = 2$.
- The existence of such a number is guaranteed by the IVT.
- The condition $f(x) = 2$ implies that $x^2 - 1 = 2$ or $x = \sqrt{3}$.

Example

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

- $f(x) = x^3 - x - 1$ is continuous everywhere, particularly on the interval $[1, 2]$.
- $f(1) = -1 < 0$.
- $f(2) = 5 > 0$.
- By IVT there is a zero of f between 1 and 2.

Continuity

Example

Consider the function

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases},$$

- $f(x)$ is defined on the closed interval $[0, 2]$.
- $f(0) = -1 < 0$.
- $f(2) = 1 > 0$.
- Note that $\frac{1}{3}$ is between -1 and 1, but there is no number $c \in [0, 2]$ such that $f(c) = \frac{1}{3}$.
- This does not contradict the IVT since f is not continuous at $x = 1$.

Continuity

Example

Show that the graph of $y = \sqrt{2x+5}$ and $y = 4 - x^2$ have an intersection point.

- We need to show that $\sqrt{2x+5} = 4 - x^2$ has a solution.
- Or $\sqrt{2x+5} + x^2 - 4 = 0$ has a root.
- Let $f(x) = \sqrt{2x+5} + x^2 - 4$.
- $f(0) = \sqrt{5} - 4 < 0$ and $f(2) = 3 > 0$.
- Note that $y_0 = 0$ is between $f(0)$ & $f(2)$ and f is continuous on $[0, 2]$.
- By IVT there is a $c \in [0, 2]$ such that $f(c) = 0$.
- Therefore, the two graphs intersect at $x = c$.

Limits Involving Infinity; Asymptotes of Graphs

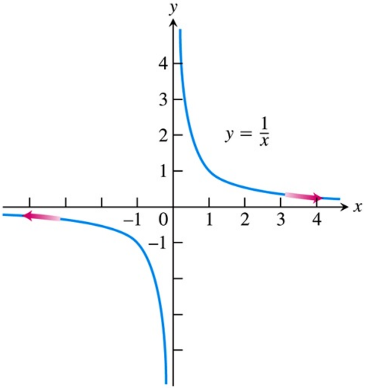
Objective

1. To define the limit as x gets infinitely large or infinitely small.
2. To find horizontal and vertical asymptotes for a given function.
3. To derive a general rule to compute horizontal asymptotes for rational functions.
4. To determine limits at the infinity for polynomials and for indeterminate cases.

Limits Involving Infinity; Asymptotes of Graphs

Example

- The function $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$.
- $\lim_{x\to\infty} f(x) = 0$.
- $\lim_{x\to-\infty} f(x) = 0$.



Limits Involving Infinity; Asymptotes of Graphs

Definition

1. We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L$$

   if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x\to-\infty} f(x) = L$$

   if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

Limits Involving Infinity; Asymptotes of Graphs

Intuitively,

- $\lim_{x\to\infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L.

- $\lim_{x\to-\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

Limits Involving Infinity; Asymptotes of Graphs

Example

Show that

1. $\lim_{x\to\infty} \frac{1}{x} = 0$.
   - Given $\epsilon > 0$, we want to find M such that for all x
     $$x > M \quad \Longrightarrow \quad |f(x) - 0| < \epsilon.$$
   - $\left|\frac{1}{x} - 0\right| < \epsilon$ iff $-\frac{1}{\epsilon} < x < \frac{1}{\epsilon}$.
   - Taking $M = \frac{1}{\epsilon}$ (or any larger positive number) proves that $\lim_{x\to\infty} \frac{1}{x} = 0$.
2. $\lim_{x\to-\infty} \frac{1}{x} = 0$.
   - Given $\epsilon > 0$, we want to find N such that for all x
     $$x < N \quad \Longrightarrow \quad |f(x) - 0| < \epsilon.$$
   - $\left|\frac{1}{x} - 0\right| < \epsilon$ iff $-\frac{1}{\epsilon} < x < \frac{1}{\epsilon}$.
   - Taking $N = -\frac{1}{\epsilon}$ (or any number less than $-\frac{1}{\epsilon}$) proves that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Limits Involving Infinity; Asymptotes of Graphs

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

Example

Evaluate the following limits:

1. $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$.

2. $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2}$

Limits Involving Infinity; Asymptotes of Graphs

Remark

To evaluate the limit of a rational function as $x \to \pm\infty$ we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

Example

Evaluate the following limits:

1. $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$. **Ans.** $\frac{5}{3}$

2. $\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$. **Ans.** 0

Limits Involving Infinity; Asymptotes of Graphs

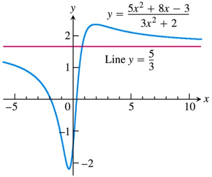
Definition

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Example

Consider the function $f(x) = \frac{5x^2+8x-3}{3x^2+2}$.



- The graph of $f(x)$ has the line $y = \frac{5}{3}$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \to \infty} = \frac{5}{3} \quad \text{and} \quad \lim_{x \to -\infty} = \frac{5}{3}$$

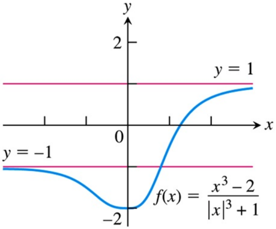
Limits Involving Infinity; Asymptotes of Graphs

Example

Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

- $\lim_{x \to \infty} f(x) = 1$.
- $\lim_{x \to \infty} f(x) = -1$.
- The horizontal asymptotes are $y = \pm 1$.



Limits Involving Infinity; Asymptotes of Graphs

Example

1. The function $f(x) = \frac{x+2}{x}$ has a horizontal asymptote $y = 1$ because $\lim_{x\to\infty} f(x) = 1$.
2. The function $f(x) = e^x$ has a horizontal asymptote $y = 0$ because $\lim_{x\to\infty} f(x) = 0$.
3. The function $f(x) = \sin\frac{1}{x}$ has a horizontal asymptote $y = 0$ because $\lim_{x\to-\infty} f(x) = 0$.
4. The function $f(x) = x\sin\frac{1}{x}$ has a horizontal asymptote $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$.
5. The function $f(x) = \tan^{-1} x$ has two horizontal asymptotes, $y = \pm\frac{\pi}{2}$ because $\lim_{x\to\pm\infty} f(x) = 1$.

Example

Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$f(x) = 2 + \frac{\sin x}{x}.$$

- We are interested in the behavior as $x \to \pm\infty$.
- $-1 \leq \sin x \leq 1 \Longrightarrow 0 \leq |\sin x| \leq 1 \Longrightarrow 0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$.
- $\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$.
- By the Sandwich Theorem $\lim_{x\to\pm\infty} \left|\frac{\sin x}{x}\right| = 0$.
- Therefore, $\lim_{x\to\pm\infty} 2 + \frac{\sin x}{x} = 2$.
- So, the line $y = 2$ is a horizontal asymptote of the curve on both left and right.