The Normal Distribution

Objectives:

After studying this chapter, the student will:

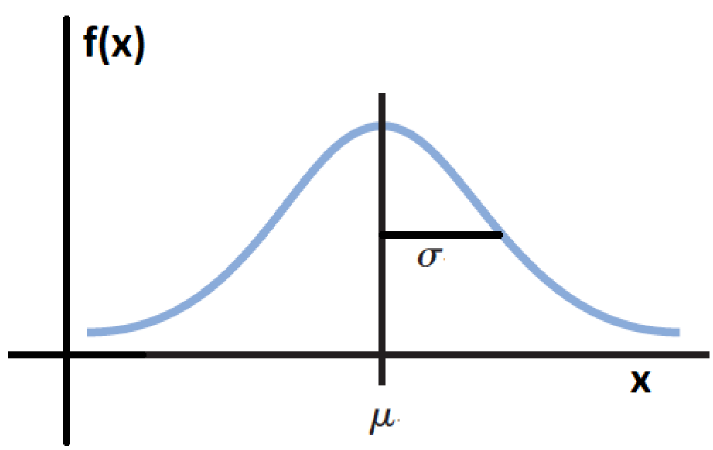
1. Identify the properties of a normal distribution.
2. Find the area under the standard normal distribution, given various z values.
3. Find probabilities for a normally distributed variable by transforming it into a standard normal variable.
4. Find specific data values for given percentages, using the standard normal distribution.

Introduction:

- Medical researchers have determined so-called normal intervals for a person's blood pressure, cholesterol, triglycerides, and the like.
- For example, the normal range of systolic blood pressure is 110 to 140. The normal interval for a person's triglycerides is from 30 to 200 milligrams per deciliter (mg/dl).
- By measuring these variables, a physician can determine if a patient's vital statistics are within the normal interval or if some type of treatment is needed to correct a condition and avoid future illnesses.
- The question then is, how does one determine the so-called normal intervals?
- In this chapter, you will learn how researchers determine normal intervals for specific medical tests by using a **normal distribution**.
- The **normal distribution** is the most important distribution in biostatistics. It is frequently called the Gaussian distribution.
- The mathematical formula for a **normal** distribution is:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

- The graph has a familiar bell-shaped curve.



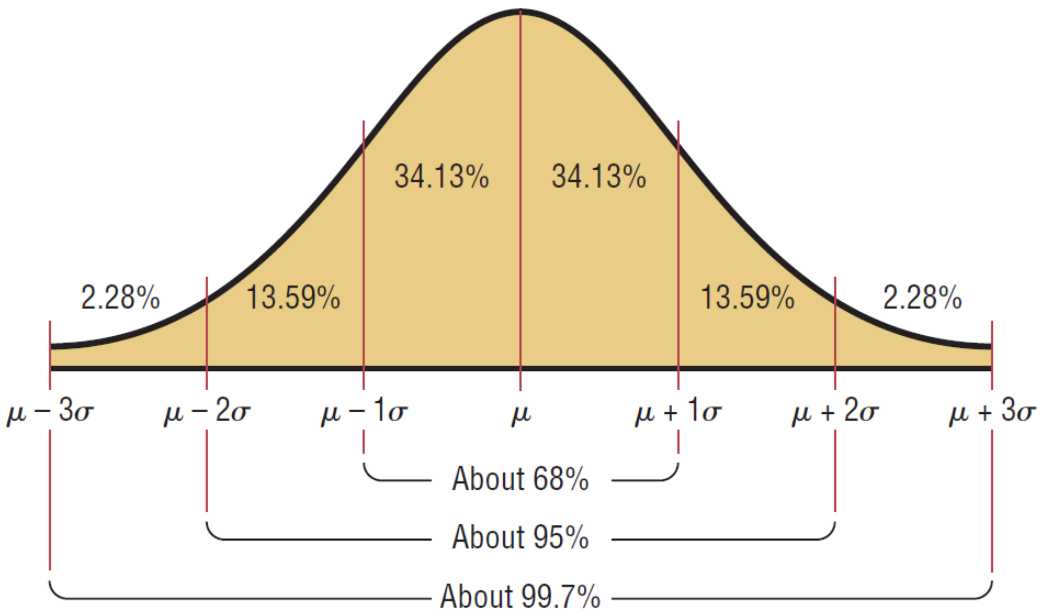
Characteristics of the normal distribution:

1. The mean, median, and mode are equal.

2. The normal curve is bell shaped and is symmetric about the mean.

3. The total area under the normal curve is equal to one. Because of the symmetry already mentioned, 50 percent of the area is to the right of μ, and 50 percent is to the left.

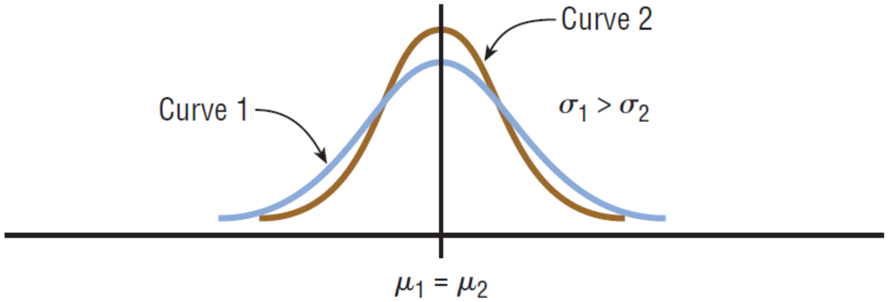
4. The area under the part of a normal curve that lies within 1 standard deviation of the mean is approximately 0.68, or 68%; within 2 standard deviations, about 0.95, or 95%; and within 3 standard deviations, about 0.997, or 99.7%.



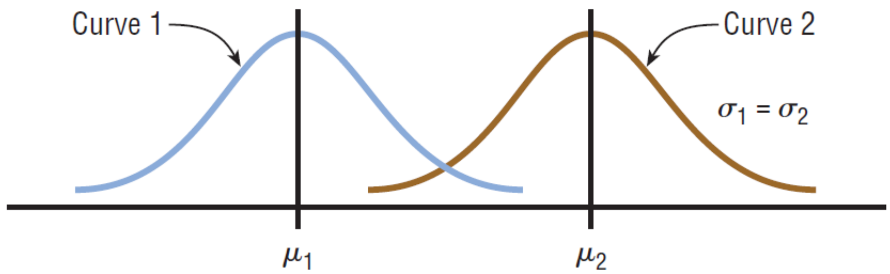
5. Between $\mu - \sigma$ and $\mu + \sigma$ (in the center of the curve) the graph curves downward. The graph curves upward to the left of $\mu - \sigma$ and to the right of $\mu + \sigma$. The points at which the curve changes from curving upward to curving downward are called inflection points.

6. The normal curve approaches, but never touches, the x-axis as it extends farther and farther away from the mean.

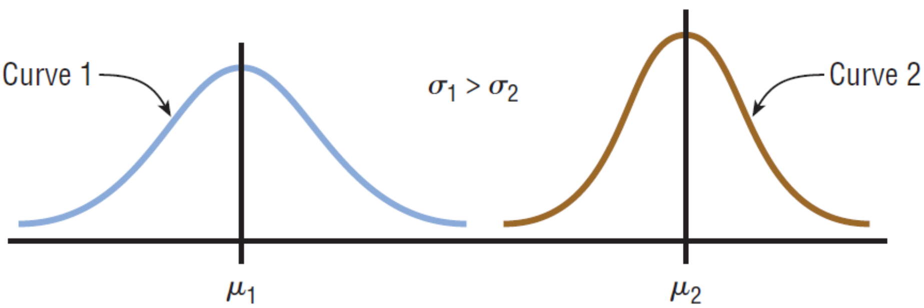
7. The normal distribution is determined by two parameters: the mean μ (location parameter) and the variance σ^2 (scale parameter).



(a) Same means but different standard deviations



(b) Different means but same standard deviations

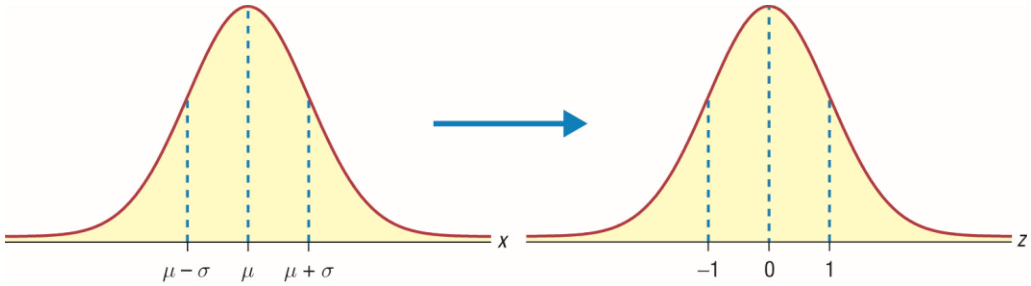


(c) Different means and different standard deviations

The Standard Normal Distribution:

- There are infinitely many normal distributions, each with its own mean and standard deviation.
- The normal distribution with a mean of 0 and a standard deviation of 1 is called the **standard normal distribution**.
- The horizontal scale of the graph of the standard normal distribution corresponds to z-score.
- Recall that a z-score is a measure of position that indicates the number of standard deviations a value lies from the mean.

- Also, recall that you can transform an x-value to a z-score using the formula $z = \frac{x-\mu}{\sigma}$ (Round to the nearest hundredth).

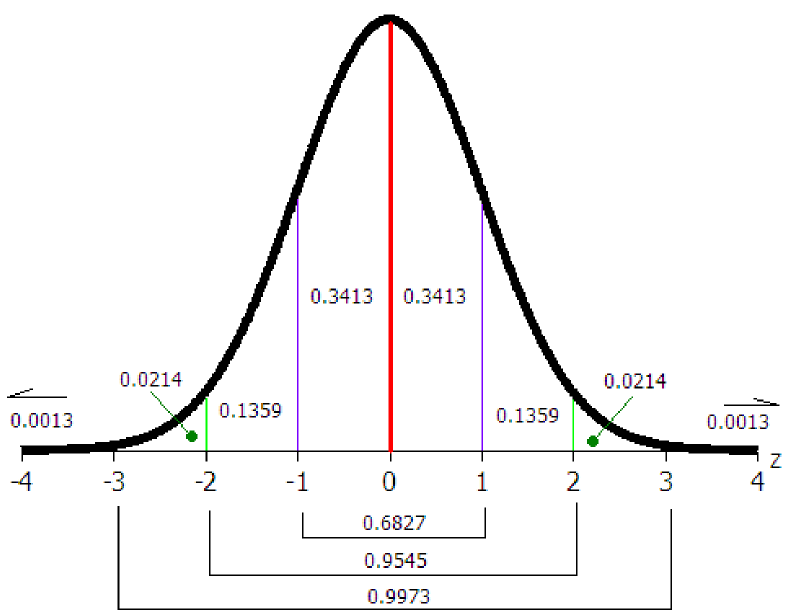


- The mathematical formula for the **standard normal** distribution is:

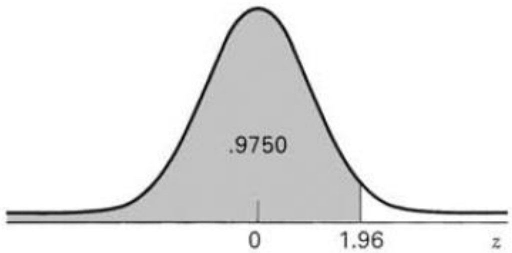
$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\}$$

Areas under the standard normal curve:

- The area under the standard normal curve that lies within 1 standard deviation of the mean is approximately 0.68, or 68%; within 2 standard deviations, about 0.95, or 95%; and within 3 standard deviations, about 0.997, or 99.7%.



- Using the standard normal table: The following table gives the area to the left of a positive z-score.



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	*z*
0.00	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	0.00
0.10	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	0.10
0.20	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	0.20
0.30	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	0.30
0.40	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	0.40
0.50	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	0.50
0.60	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	0.60
0.70	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852	0.70
0.80	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	0.80
0.90	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	0.90
1.00	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	1.00
1.10	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	1.10
1.20	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	1.20
1.30	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	1.30
1.40	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1.40
1.50	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1.50
1.60	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1.60
1.70	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1.70
1.80	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1.80
1.90	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1.90
2.00	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	2.00
2.10	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	2.10
2.20	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	2.20
2.30	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	2.30
2.40	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	2.40
2.50	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	2.50
2.60	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	2.60
2.70	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	2.70
2.80	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	2.80
2.90	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	2.90
3.00	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	3.00
3.10	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	3.10
3.20	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	3.20
3.30	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	3.30
3.40	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	3.40
3.50	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	3.50
3.60	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	3.60
3.70	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	3.70
3.80	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	3.80

Example:

Find the area under the standard normal curve to the left of $z = 1.96$.

Solution:

Example:

Find the area under the standard normal curve to the right of $z = 1.96$.

Solution:

Example:

Find the area under the standard normal curve from $z = -2.57$ to $z = 0$.

Solution:

Example:

Find the area under the standard normal curve from $z = 0$ to $z = 1.23$.

Solution:

Example:

Find the area under the standard normal curve from $z = -2.57$ to $z = 1.23$.

Solution:

Example:

Find the following probabilities for the standard normal curve.

1. $P(Z < 2.33)$.
2. $P(Z > -2.33)$.
3. $P(-1.23 < Z < 1.23)$.
4. $P(-1.23 < Z < 2.75)$.
5. $P(1.23 < Z < 2.75)$.
6. $P(-2.75 < Z < -1.23)$.
7. $P(Z > 3.98)$.

Solution:

Finding the z-score when the area is given:

Example:

Given the following probabilities, find z_1.

1. $P(Z < z_1) = 0.0055$.
2. $P(Z > z_1) = 0.0384$.
3. $P(z_1 < Z < 2.98) = 0.1117$.

Solution:

Example:

Find z_1 for which 95% of the distribution's area lies between $-z_1$ and z_1.

Solution:

Example:

Find z-score that corresponds to percentile P_{95}.

Solution:

Normal Distribution Applications:

To find areas and probabilities for a random variable X that follows a normal distribution with mean μ and standard deviation σ, convert x values to z values using the formula

$$z = \frac{x - \mu}{\sigma}$$

Then use the standard normal table to find corresponding areas and probabilities.

Example:

A survey found that women spend on average $146.21 on beauty products during the summer months. Assume the standard deviation is $29.44. Find the percentage of women who spend less than $160.00. Assume the variable is normally distributed.

Solution:

Example:

The systolic blood pressure of 16 middle age men before exercise has a Normal distribution with a mean of 141.1 mmHg and a standard deviation of 13.62 mmHg. What is the probability having a systolic blood pressure of 160 mmHg or higher?

Solution:

Example:

The average number of calories in a 1.5-ounce chocolate bar is 225. Suppose that the distribution of calories is approximately normal with $\sigma = 10$. Find the probability that a randomly selected

chocolate bar will have between 200 and 220 calories.

Solution:

Example:

Americans consume an average of 1.64 cups of coffee per day. Assume the variable is approximately normally distributed with a standard deviation of 0.24 cup. If 500 individuals are selected, approximately how many will drink less than 1 cup of coffee per day?

Solution:

Finding the x-value given specific probability:

Example:

In a randomly selected sample of 1169 men ages 35-44, the mean total cholesterol level was 205 milligrams per deciliter with a standard deviation of 39.2 milligrams per deciliter. Assume the total cholesterol levels are normally distributed. Find the highest total cholesterol level a man in this 35-44 age group can have and be in the lowest 1%.

Solution:

Example:

Scores for a biostatistics exam are normally distributed, with a mean of 75 and a standard deviation of 6.5. To be eligible for medical school, you must score in the top 5%. What is the lowest score you can earn and still be eligible?

Solution:

Example:

For a medical study, a researcher wishes to select people in the middle 60% of the population based on blood pressure. If the mean systolic blood pressure is 120 and the standard deviation is 8, find the upper and lower readings that would qualify people to participate in the study.

Solution:

Exercises:

Q1. Use the Standard Normal Table to find the indicated area under the standard normal curve.

1. To the left of $z = -0.84$.
2. To the left of $z = 2.55$.
3. To the right of $z = 1.68$.
4. To the right of $z = -0.83$.
5. Between $z = -1.64$ and the mean.
6. Between $z = -1.22$ and $z = -0.43$.
7. To the left of $z = 0.12$ and the right of $z = 1.72$.

Q2. If the total cholesterol values for a certain target population are approximately normally distributed with a mean of 200 (mg/100 mL) and a standard deviation of 20 (mg/100 mL). Find the probability that a person picked at random from this population will have a cholesterol value greater than 240 (mg/100 mL).

Q3. A psychologist has devised a stress test for dental patients sitting in the waiting rooms. According to this test, the stress scores (on a scale of 1 to 10) for patients waiting for root canal treatments are found to be approximately normally distributed with a mean of 7.59 and a standard deviation of .73.

1. What percentage of such patients have a stress score lower than 6.0?
2. What is the probability that a randomly selected root canal patient sitting in the waiting room has a stress score between 7.0 and 8.0?
3. The psychologist suggests that any patient with a stress score of 9.0 or higher should be given a sedative prior to treatment. What percentage of patients waiting for root canal treatments would need a sedative if this suggestion is accepted?

Q4. The average length of a hospital stay for all diagnoses is 4.8 days. If we assume that the lengths of hospital stays are normally distributed with a variance of 2.1, then 10% of hospital stays are longer than how many days? Thirty percent of stays are less than how many days?

Q5. The average per capita spending on health care in the United States is $5274. If the standard deviation is $600 and the distribution of health care spending is approximately normal, what is the probability that a randomly selected person spends more than $6000? Find the limits of the middle 50% of individual health care expenditures.

Q6. In a large section of a statistics class, the points for the final exam are normally distributed with mean of 72 and a standard deviation of 9. Grades are to be assigned according to the following rule.

The top 10% receive A′s.

The next 20% receive B′s.

The middle 40% receive C′s.

The next 20% receive D′s.

The bottom 10% receive F′s.

Find the lowest score on the final exam that would qualify a student for an A, a B, a C, and a D.

Q7. The speeds of vehicles along a stretch of highway are normally distributed, with a mean of 56 miles per hour and a standard deviation of 4 miles per hour. Find the speeds x corresponding

to z-scores of 1.95, -2.33, and 0.

Q8. In the study of fingerprints, an important quantitative characteristic is the total ridge count for the 10 fingers of an individual. Suppose that the total ridge counts of individuals in a certain population are approximately normally distributed with a mean of 140 and a standard deviation of 50. Find the probability that an individual picked at random from this population will have a ridge count of:

1. 200 or more

2. Less than 100

3. Between 100 and 200

4. Between 200 and 250

5. In a population of 10,000 people how many would you expect to have a ridge count of 200 or more?

Q9. The IQs of individuals admitted to a state school for the mentally retarded are approximately normally distributed with a mean of 60 and a standard deviation of 10.

1. Find the proportion of individuals with IQs greater than 75.

2. What is the probability that an individual picked at random will have an IQ between 55 and 75?